THE UNIVERSITY OF WESTERN AUSTRALIA
Achieving International Excellence

Modeling Complex Systems
Recurence Equations

Farid, csse building, room 1.05

consultation hour: Tuesday 1:00 pm – 2:00pm

Outlines

- Definition
- Iteration of maps
- Graphical analysis
- Stability
- Examples

Definition

- We consider recurrence equations of the form:

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t)$$

- x, representing the state of the system, belongs to a subset *J of* R^n,
- f : $J \rightarrow J$ *is a* ***map*** *(function),* $t \in N_0=\{0,1,2,3,\ldots\}$

Iteration of maps

- We adopt the notation:

$$\mathbf{f}^{t+1} = \mathbf{f} \circ \mathbf{f}^{t}, \quad \text{where} \quad \mathbf{f}^{1} = \mathbf{f}$$

where $t \in N_0$ (f^0 is the identity)

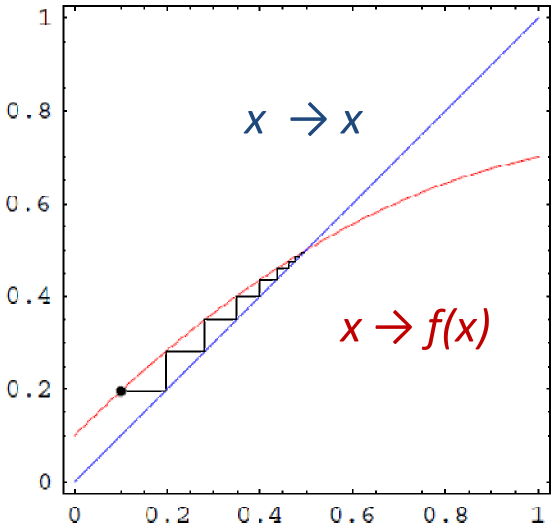
- In the case of maps, the analogue of the flow (defined for differential equations) is the mapping $\mathbf{f}^{t}: J \rightarrow J$, defined for $t \in N_0$
- The forward **orbit of f through x** = $\{f^t(x) \mid t \in N_0\}$ oriented in the sense of increasing t

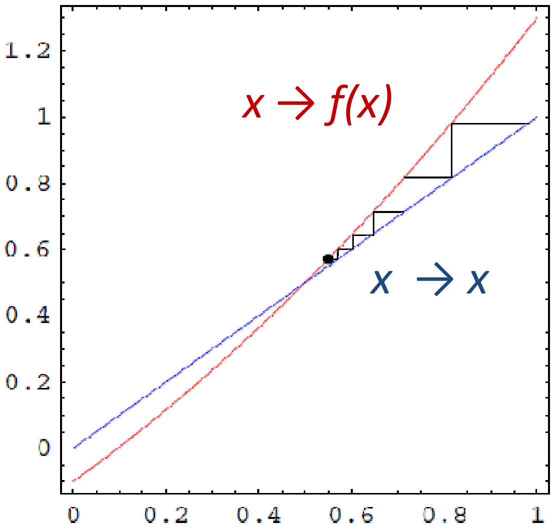
Iteration of maps

- A point $x^* \in J$ *is an* ***equilibrium point*** *or a* ***fixed point*** *if* $f(x^*) = x^*$. → The orbit of an equilibrium point is the equilibrium point itself.
- A point $x^* \in J$ *is* a *periodic point if* $\mathbf{f^{\tau}(x^*) = x^*}$ *for some nonzero* τ *. The smallest positive value* of τ *is the period of the point and is usually denoted by T.*
- *A periodic point* of period T *is a fixed point of* $\mathbf{f^T}$ **.**
- *The set* $\{f^t(x^*) \mid t = 0, 1, \dots, T - 1\}$ *is a periodic orbit or a T-point cycle. A periodic orbit consists of a finite number* of points.

Graphical analysis

- In the case of one-dimensional maps, there exists a simple graphical method to follow the successive iterates of an initial point x_0





- Cobwebs in the vicinity of equilibrium points. The equilibrium point in the left diagram is asymptotically stable ($|f'(x^*)| < 1$), while the equilibrium point in the right diagram is unstable ($|f'(x^*)| > 1$). Dots represent initial values.

Stability 1

- A fixed point x^* of the recurrence equation $x_{t+1} = f(x_t)$ is Lyapunov stable if, for any given positive ε, there exists a positive δ (depends on ε only) such that $|f^t(x) - x^*| < \varepsilon$ for all x such that $| x - x^* | < \delta$ and all $t > 0$.
- A fixed point x^* is asymptotically stable if it is Lyapunov stable and if there exists a positive δ such that, for all x such that $| x - x^* | < \delta$, $\lim_{t\to\infty} f^t(x) = x^*$.
- Fixed points that are not stable *are unstable.*

- In other words: the iterates of points close to a Lyapunov stable fixed point remain close to it, while the iterates of points close to an asymptotically stable fixed point move towards it as *t increases.*

Stability 2

- If a metric d is defined on the phase space J, a map **$f : J \rightarrow J$** is said to be **contracting** if there exists a positive real **$\lambda < 1$** such that, for any pair $(x, y) \in J \times J$**: $d(f(x), f(y)) \leq \lambda d(x, y)$.**

- f a contracting map → $(f^t(x))_{t \in N}$ converges exponentially to the unique fixed point x^* of f .

Stability 3

- In the case of linear maps, if all the eigenvalues of a linear map *A :* $R_n \rightarrow R_n$ *have absolute* values less than one, the map is contracting and the iterates of every point converge to the origin exponentially

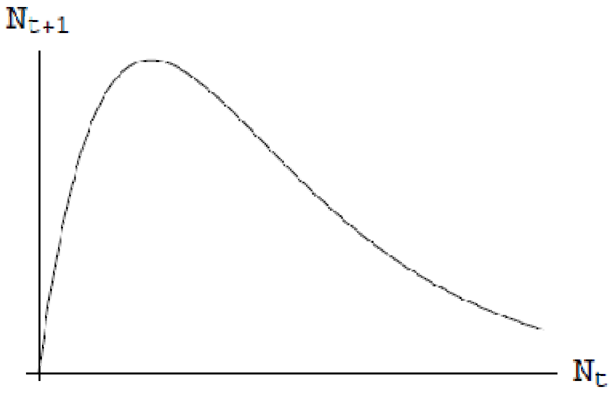
- *A fixed point* $x^* \in S$ *is said to be hyperbolic if none of the eigenvalues of the Jacobian matrix* $Df(x*)$ *has modulus equal to one. The linear function* $x \rightarrow Df(x^*)x$ *is called the linear part of f at* x^**.*

Example1: o*ne-population models*

- *Assymption: a species breeds only at a particular* time of the year.

$$N_{t+1} = f(N_t).$$

- *f should satisfy* the following two conditions:
  1. *the image f(N) of any positive N should be positive,*
  2. *f should be increasing for small N and decreasing for large N.*



Example1: *one-population models*

- *1st model:* $N_{t+1} = N_t \exp\left(r\left(1 - \frac{N_t}{K}\right)\right)$, *r and K > 0*

→ $n_{t+1} = n_t \exp\left(r(1 - n_t)\right)$; $n_t = N_t/K$

- Two equilibrium points 0 and 1.
- The values of the derivative of the function *n → n exp(r(1 − n)) at n = 0 and n = 1 are, respectively,* e^r and $1 - r$*, 0 is always unstable and 1 is asymptotically stable if* $0 < r < 2$.

Example1: *one-population models*

- *2nd model:* $N_{t+1} = \dfrac{rN_t}{\left(1 + \dfrac{N_t}{K}\right)^a}$ *; r, K and a > 1*

→ $n_{t+1} = \dfrac{rn_t}{(1+n_t)^a}$; $n_t = N_t/K$

- Two equilibrium points 0 and $r^{1/a} - 1$.
- The derivative of the function *n → rn/(1 + n)a at n = 0 and n =* $r^{1/a} - 1$ *are,* respectively, *r and* $1 - a(1 - r^{-1/a})$*, 0 is always unstable and r1/a − 1 is* asymptotically stable if $0 < a(1 - r^{-1/a}) < 2$.

Example1: *one-population models*

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→ $$n_{t+1} = \frac{rn_t}{(1+n_t)^a}$$ *;* $n_t = N_t/K$

- Two equilibrium points 0 and $r^{1/a} - 1$.
- The derivative of the function $n \rightarrow rn/(1 + n)a$ *at* $n = 0$ *and* $n = r^{1/a} - 1$ *are,* respectively, r *and* $1 - a(1 - r^{-1/a})$*, 0 is always unstable and* $r1/a - 1$ *is* asymptotically stable if $0 < a(1 - r^{-1/a}) < 2$.