THE UNIVERSITY OF WESTERN AUSTRALIA
Achieving International Excellence

Modeling Complex Systems
Bifurcation

Farid, CSSE building, room 1.05

Consultation: Tuesday 1:00 pm – 3:00pm

Outlines

- What is bifurcation?
- Bifurcation in Differential Equation (DE) models
- Example
- Bifurcation in Recurrence Equation (RE) models
- Example
- Project II

What is Bifurcation?

- Study of dynamical systems: a **bifurcation** occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior.
- DE: change in a family of vector fields $dx/dt=f(x,\lambda)$.
- RE: Change in Maps $x_{t+1}=f(x_t,\lambda)$.

Bifurcation in Differential Equation (DE) models

- If at the equilibrium point (x^*, μ^*) *some eigenvalues* of *$DX(x^*, \mu^*)$ have zero real part, a variation of μ close to μ^* gives a totally new dynamical* behavior of the system.
- We will see the simplest bifurcations that occur for 1D and 2D system.

Bifurcation in DE models saddle-node bifurcation

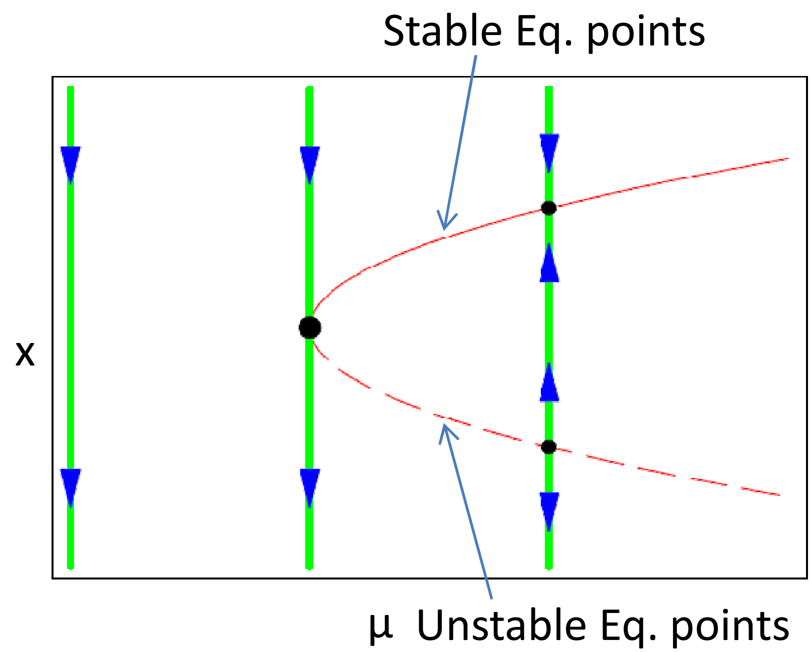
- In saddle-node bifurcation or tangential bifurcation **two fixed points** of a the system **collide and annihilate each other**. One of the equilibrium points is unstable (the saddle), while the other is stable (the node).
- Example: $\dot{x} = \mu - x^2$
- $\mu=0$, Eq. point= 0; Dx(0,0)=0;
- $\mu<0$, no Eq. point;
- $\mu>0$, two Eq. points=±sqrt(μ):

  +sqrt(μ) stable , -sqrt(μ) unstable

Bifurcation in DE models
saddle-node bifurcation

- *Bifurcation diagrams with phase portraits of the system:*

$$\dot{x} = \mu - x^2$$



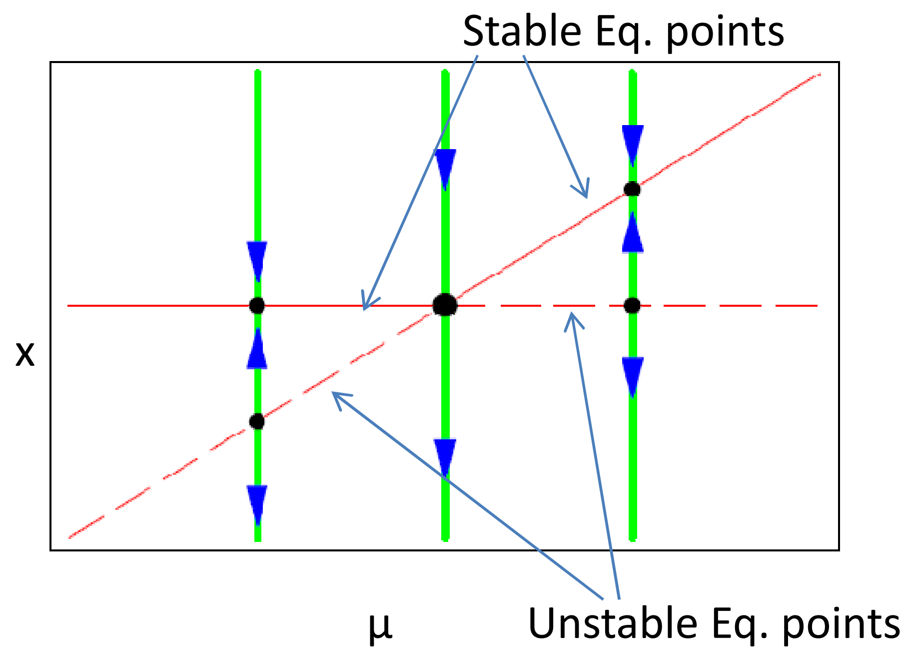
Bifurcation in DE models
Transcritical bifurcation

- In transcritical bifurcation, before and after the bifurcation, there are **one unstable and one stable fixed points**. However, **their stability is exchanged** when they collide. So the unstable fixed point becomes stable and vice versa.
- Example: $\dot{x} = \mu x - x^2$
- $\mu=0$, Eq. point= 0; Dx(0,0)=0;
- $\mu \neq 0$, two Eq. points=μ and 0;

Bifurcation in DE models
Transcritical bifurcation

- *Bifurcation diagrams with phase portraits of the system:*

$$\dot{x} = \mu x - x^2$$



Bifurcation in DE models Pitchfork bifurcation

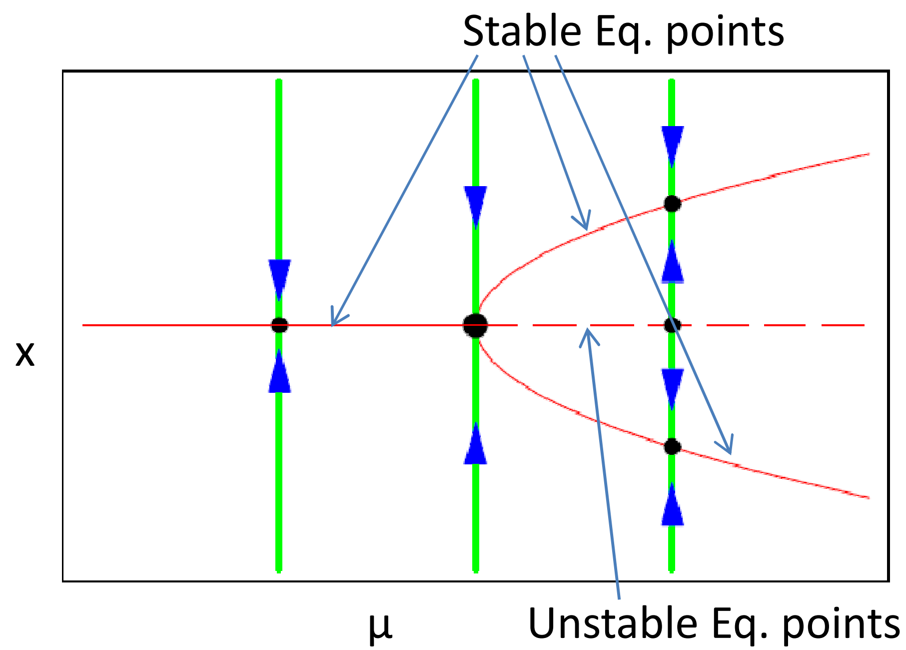
- In Pitchfork bifurcation, there are **two stable and one unstable fixed points before the bifurcation**, and **one stable fixed point after the bifurcation**. At the bifurcation point, the **stable points collide and annihilate each other**, and the **unstable point becomes stable**.
- Example: $\dot{x} = \mu x - x^3$.
- μ≤0, Eq. point=0; stable
- μ>0, three Eq. points=0 and ±sqrt(μ):

  ± sqrt(μ) stable , 0 unstable

Bifurcation in DE models
Pitchfork bifurcation

- *Bifurcation diagrams with phase portraits of the system:*

$$\dot{x} = \mu x - x^3.$$



Bifurcation in DE models
Hopf bifurcation: Example

- In Hopf bifurcation, a fixed point loses stability as **a pair of complex conjugate eigenvalues** of the linearisation around the fixed point **cross the imaginary axis of the complex plane.**

Bifurcation in DE models
Hopf bifurcation: Example

- Harrison's model:

$$\dot{H} = r_H H\left(1 - \frac{H}{K}\right) - \frac{a_H P H}{b + H}$$

$$\dot{P} = \frac{a_P P H}{b + H} - cP,$$

- Trivial eq. points (0,0) and (K,0). Non trivial one (H^*, P^*).

$$h = \frac{H}{H^*}, \quad p = \frac{P}{P^*}, \quad \tau = r_H t, \quad k = \frac{K}{H^*}, \quad \beta = \frac{b}{H^*}, \quad \gamma = \frac{c}{r}.$$

$$\frac{dh}{d\tau} = h\left(1 - \frac{h}{k}\right) - \frac{\alpha_h p h}{\beta + h}, \qquad \alpha_h = \left(1 - \frac{1}{k}\right)(\beta + 1)$$

$$\frac{dp}{d\tau} = \frac{\alpha_p p h}{\beta + h} - \gamma p, \qquad \alpha_p = \gamma(\beta + 1)$$

Bifurcation in DE models
Hopf bifurcation: Example

- $(h^*, p^*)=(1,1)$.

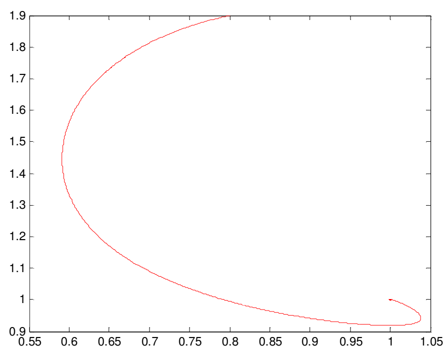
$$Df(1,1) = \begin{bmatrix} \frac{k-2-\beta}{k(1+\beta)} & -1+\frac{1}{k} \\ \frac{\beta\gamma}{1+\beta} & 0 \end{bmatrix}.$$

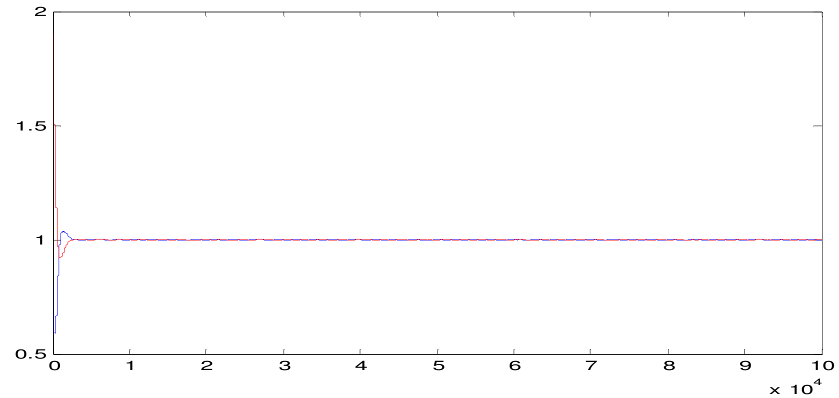
- The eigenvalues:

$$\frac{k-2-\beta \pm \mathrm{i}\sqrt{4(k-1)k^2\beta\gamma-(k-2-\beta)^2}}{2k(1+\beta)}.$$

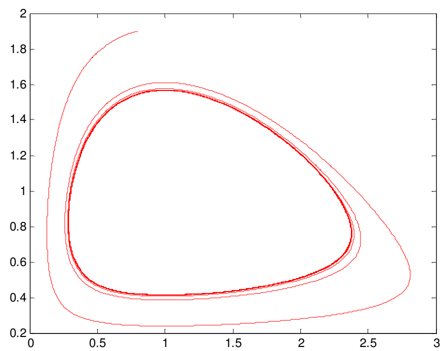
- $k < 2+\beta$ → real parts are negative → Asympt. Stable.
- $k > 2+\beta$ → real parts are positive → Unstable

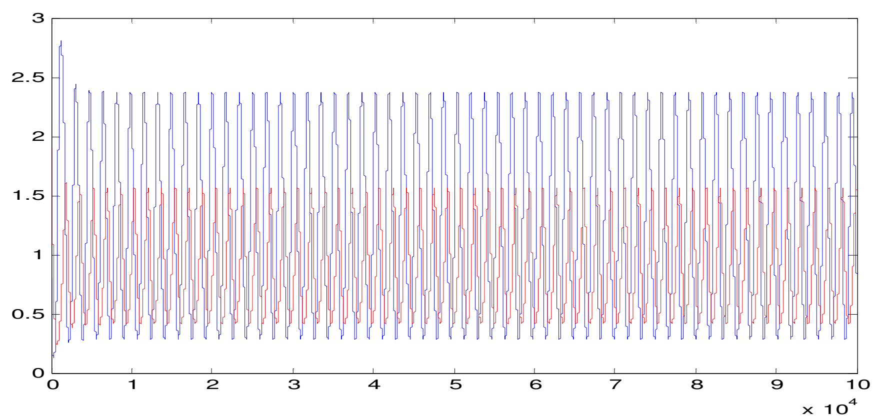
Bifurcation in DE models
Hopf bifurcation: Example





K=1.5





K=3.5

Phase portraits in (Prey,Predator) plan

Evolutions of Prey and Predator populations with time

Bifurcation in Recurrence Equation (RE) models

- If at the equilibrium point (x^*, μ^*) *some eigenvalues* of *$DX(x^*, \mu^*)$ have their modulus equal to 1, a variation of μ close to μ^* gives a totally new dynamical* behavior of the system.
- We will see the simplest bifurcations that occur for 1D and 2D system.

Bifurcation in Recurrence Equation (RE) models

- Saddle-node bifurcation: eg. $x_{t+1} = x_t + \mu - x_t^2$
- Transcritical bifurcation: eg. $x_{t+1} = x_t + \mu x_t - x_t^2$
- Pitchfork bifurcation: eg. $x_{t+1} = x_t + \mu x_t - x_t^3$
- Hopf bifurcation: (a pair of complex conjugate eigenvalues cross the unit circle).

example: *$n_{t+1} = rn_t(1 - n_{t-1})$; if $x_t = n_{t-1}$ and $y_t = n_t$, a 2D system is obtained:*

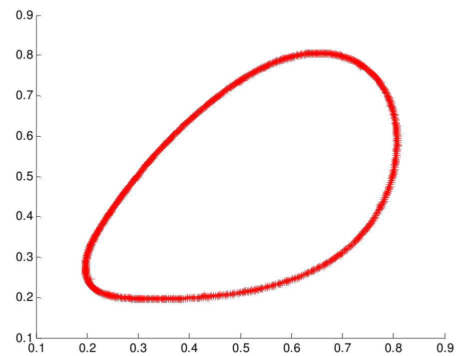
$$x_{t+1} = y_t \qquad y_{t+1} = r\, y_t\,(1 - x_t)$$

Bifurcation in RE models
Hopf bifurcation, Example

- Two fixed points: (0,0) and (1-1/r,1-1/r)
- The eigenvalues of Df((1-1/r,1-1/r), r) are:

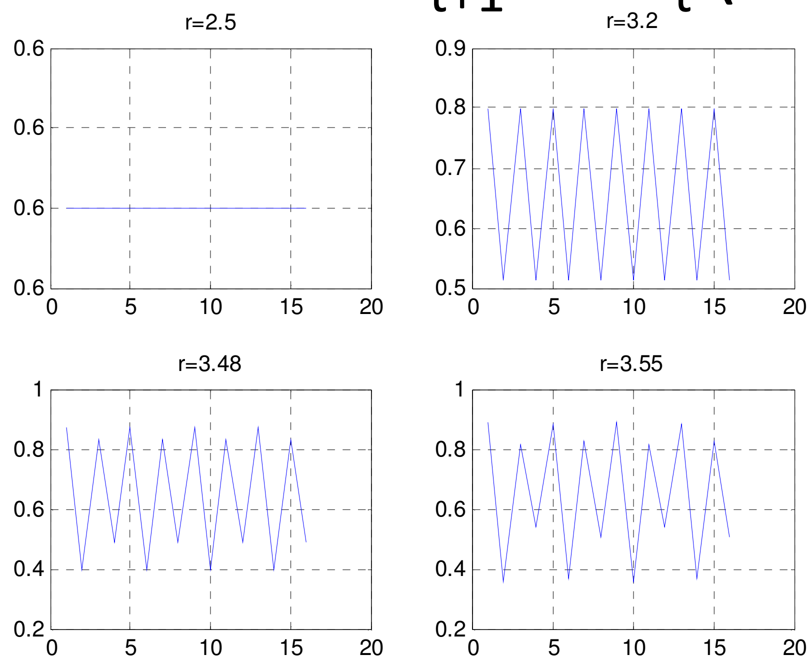
½ (1 ±sqrt(5-4r))

- R>4/5 → two complex conjugate eigenvalues. Their modulus is equal to 1 when r=2 (bifurcation point).
- Phase portraits of the delayed logistic model, r=2.1



Bifurcation in RE models
Period doubling

- In **Period doubling bifurcation** a discrete dynamical system **switches to a new behavior with twice the period** of the original system.
- The logistic model: $n_{t+1} = r\, n_t\, (1- n_t)$



Bifurcation in RE models
Hopf bifurcation, Example