# Computer Vision CITS4240

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# Mathematical Morphology

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Binary images in general will contain more than one object. In order to identify or classify the objects in the image, we must first identify the *connected components*, that is the distinctly connected blobs that correspond to each object in the image. At this stage, we are assuming that each object is isolated from the others, that is, there is no occlusion. The sequential scan labelling algorithm we describe comes from Horn's book [2].

We then go on to describe the technique of *mathematical morphology* for low-level image analysis. The notes in this section follow closely those of Gonzalez and Woods [1].

## Identifying connected components

Given a binary image we wish to scan through it, identify distinct 'blobs' and label each one uniquely. Connectivity will be described using a left-skewed 6-connectedness neighbourhood scheme, as shown in figure 1.

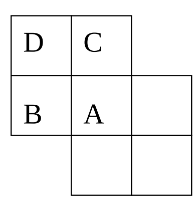


Figure 1: The 6-connected neighbourhood.

### A sequential scan labelling algorithm

We scan the image using a typical raster scan, row by row, top to bottom, left to right. Then, when we examine a particular cell $A$, we know that the cell to its left, $B$, has already been labelled, as has the cell $C$ directly above $A$. Moreover, the cell $D$ directly above $B$ is also considered connected to $A$ so its labelling must also be taken into account.

The sequential scan labelling algorithm is described as follows:

```
if A = 0          do nothing

else if D labeled
     copy label to A

else if (not B labeled) and (not C labeled)
     increment label numbering and label A

else if B xor C labeled
     copy label to A

else if B and C labeled
     if B label = C label
          copy label to A
     else
          copy either B label or C label to A
          record equivalence of labels
```

After running this algorithm to label all the pixels, a second scan through the image is required to clean up the label equivalences, giving each connected component in the image a unique label.

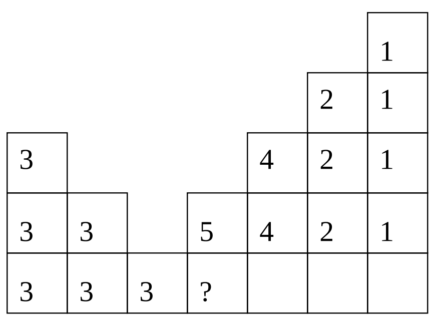


Figure 2: Two regions previously thought to be disconnected are later discovered to be connected. Equivalent labels must be identified by a second scan.

# Mathematical Morphology

Mathematical Morphology is a tool for extracting image components that are useful for representation and description. The technique was originally developed by Matheron and Serra [3] at the Ecole des Mines in Paris. It is a set-theoretic method of image analysis providing a quantitative description of geometrical structures. (At the Ecole des Mines they were interested in analysing geological data and the structure of materials.) Morphology can provide boundaries of objects, their skeletons, and their convex hulls. It is also useful for many pre- and post-processing techniques, especially in edge thinning and pruning.

Generally speaking most morphological operations are based on simple expanding and shrinking operations. The primary application of morphology occurs in binary images, though it is also used on grey level images. It can also be useful on range images. (A range image is one where grey levels represent the distance from the sensor to the objects in the scene rather than the intensity of light reflected from them.)

## Set operations

The two basic morphological set transformations are *erosion* and *dilation*

These transformations involve the interaction between an image $A$ (the object of interest) and a structuring set $B$, called the *structuring element.*

Typically the structuring element $B$ is a circular disc in the plane, but it can be any shape. The image and structuring element sets need not be restricted to sets in the 2D plane, but could be defined in 1, 2, 3 (or higher) dimensions.

Let $A$ and $B$ be subsets of $Z^2$. The *translation* of $A$ by $x$ is denoted $A_x$ and is defined as

$$A_x = \{c : c = a + x, \text{for } a \in A\}.$$

The *reflection* of $B$, denoted $\hat{B}$, is defined as

$$\hat{B} = \{x : x = -b, \text{for } b \in B\}.$$

The complement of $A$ is denoted $A^c$, and the difference of two sets $A$ and $B$ is denoted $A - B$.

### Dilation

Dilation of the object $A$ by the structuring element $B$ is given by

$$A \oplus B = \{x : \hat{B}_x \cap A \neq \emptyset\}.$$

The result is a new set made up of all points generated by obtaining the reflection of $B$ about its origin and then shifting this relection by $x$.

Consider the example where $A$ is a rectangle and $B$ is a disc centred on the origin. (Note that if $B$ is not centred on the origin we will get a translation of the object as well.) Since $B$ is symmetric, $\hat{B} = B$.

This definition becomes very intuitive when the structuring element $B$ is viewed as a convolution mask.

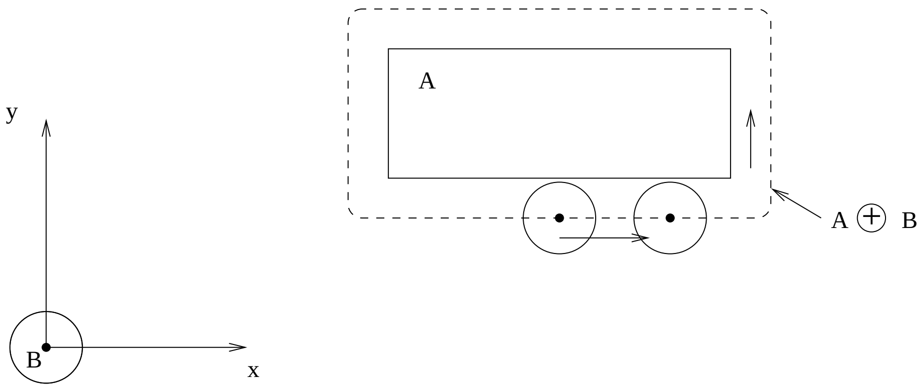


Figure 3: $A$ is dilated by the structuring element $B$.

**Erosion**

Erosion of the object $A$ by a structuring element $B$ is given by

$$A \ominus B = \{x : B_x \subseteq A\}.$$

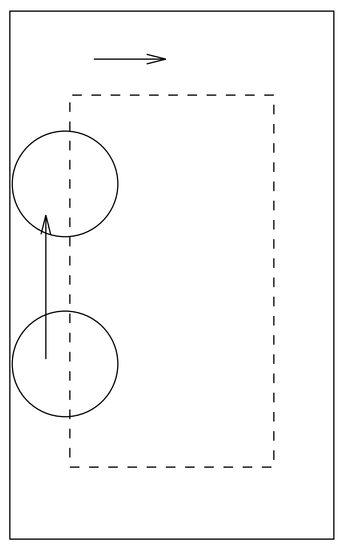

Figure 4: $A$ is eroded by the structuring element $B$ to give the internal dashed shape.

Dilation and erosion are duals of each other with respect to set complementation and reflection. That is,

$$(A \ominus B)^c = A^c \oplus \hat{B}.$$

To see this, consider first the left hand side:

$$(A \ominus B)^c = \{x : B_x \subseteq A\}^c.$$

Now, if $B_x$ is contained in $A$, then $B_x \cap A^c = \emptyset$, and so

$$(A \ominus B)^c = \{x : B_x \cap A^c = \emptyset\}^c.$$

But the complement of the set of all $x$s that satisfy $B_x \cap A^c = \emptyset$ is just the set of all $x$s such that $B_x \cap A^c \neq \emptyset$. Thus

$$(A \ominus B)^c = \{x : B_x \cap A^c \neq \emptyset\} = A^c \oplus \hat{B}.$$

**Applications of morphological operations**

Erosion and dilation can be used in a variety of ways, in parallel and series, to give other transformations including thickening, thinning, skeletonisation and many others.

Two very important transformations are *opening* and *closing*. Now intuitively, dilation expands an image object and erosion shrinks it. Opening generally smooths a contour in an image, breaking narrow isthmuses and eliminating thin protrusions. Closing tends to narrow smooth sections of contours, fusing narrow breaks and long thin gulfs, eliminating small holes, and filling gaps in contours.

The opening of $A$ by $B$, denoted by $A \circ B$, is given by the erosion by $B$, followed by the dilation by $B$, that is

$$A \circ B = (A \ominus B) \oplus B.$$

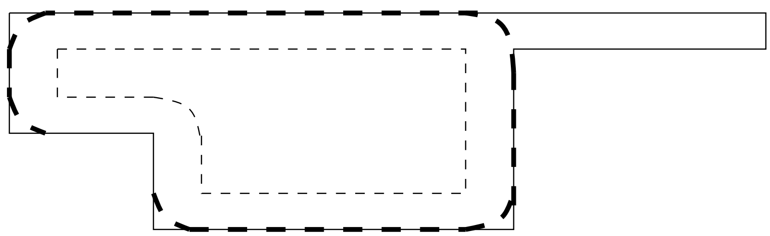

Figure 5: The opening (given by the dark dashed lines) of $A$ (given by the solid lines. The structuring element $B$ is a disc. The internal dashed structure is $A$ eroded by $B$.

Opening is like 'rounding from the inside': the opening of $A$ by $B$ is obtained by taking the union of all translates of $B$ that fit inside $A$. Parts of $A$ that are smaller than $B$ are removed. Thus

$$A \circ B = \bigcup \{B_x : B_x \subseteq A\}.$$

Closing is the dual operation of opening and is denoted by $A \bullet B$. It is produced by the dilation of $A$ by $B$, followed by the erosion by $B$:

$$A \bullet B = (A \oplus B) \ominus B.$$

This is like 'smoothing from the outside'. Holes are filled in and narrow valleys are 'closed'.

Just as with dilation and erosion, opening and closing are dual operations. That is

$$(A \bullet B)^c = (A^c \circ B^c).$$

The opening operation satisfies the following properties:

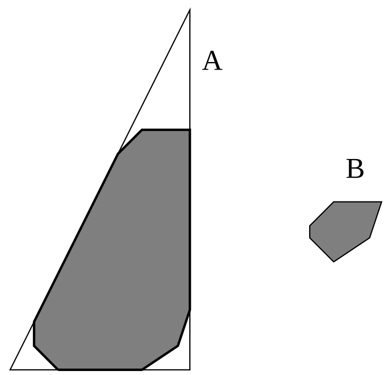


Figure 6: The opening of $A$ by the structuring element $B$.

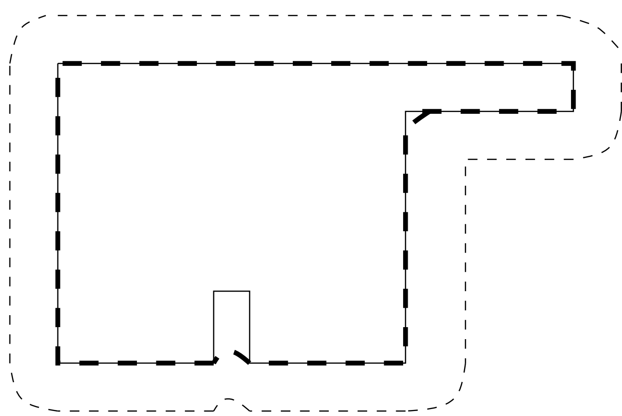

Figure 7: The closing of $A$ by the structuring element $B$.

1. $A \circ B$ is a subset of $A$.
2. If $C$ is a subset of $D$, then $C \circ B$ is a subset of $D \circ B$.
3. $(A \circ B) \circ B = A \circ B$.

Similarly

1. $A$ is a subset of $A \bullet B$.
2. If $C$ is a subset of $D$, then $C \bullet B$ is a subset of $D \bullet B$.
3. $(A \bullet B) \bullet B = A \bullet B$.

Property 3, in both cases, is known as *idempotency*. It means that any application of the operation more than once will have no further effect on the result.

The morphological filter $(A \circ B) \bullet B$ can be used to eliminate 'salt and pepper' noise. Salt and pepper noise is random, uniformly distributed small noisy elements often found corrupting real images. It will appear as black dots or small blobs on a white background, and white dots or small blobs on the black object. The background noise is eliminated at the erosion stage, under the assumption that all noise components are physically smaller than the structuring element $B$. Erosion on its own will increase the size of the noise components on the object. However, these are eliminated at the closing operation.

The important thing to note is that morphological operations preserve the main geometric structures of the object. Only features 'smaller than' the structuring element are affected by transformations. All other features at 'larger scales' are not degraded. (This is not the case with linear transformations, such as convolution).

The *boundary* of a set $A$, denoted $\partial A$, can be obtained by first eroding $A$ with $B$, where $B$ is a suitable structuring element, and then performing the set difference between $A$ and its erosion. That is

$$\partial A = A - (A \ominus B).$$

Typically, $B$ would be a $3 \times 3$ matrix of 1s.

*Region filling* can be accomplished iteratively using dilations, complementation, and intersections. Suppose we have an image $A$ containing a subset whose elements are 8-connected boundary points of a region. Beginning with a point $p$ inside the boundary, the objective is to fill the entire region with 1s.

Since, by assumption, all non-boundary points are labeled 0, we begin by assigning 1 to $p$, and then construct

$$X_k = (X_{k-1} \oplus B) \cap A^c, \text{for } k = 1, 2, \ldots$$

where $X_0 = p$, and $B$ is the 'cross' structuring element shown in figure 8. The algorithm terminates when $X_k = X_{k-1}$. The set union of $X_k$ and $A$ contains the filled set and its boundary.

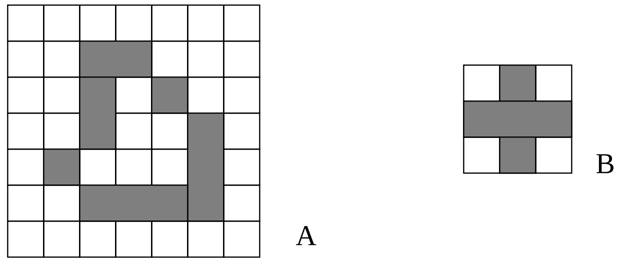

Figure 8: The region in $A$ is filled using the structuring element $B$.

Likewise, *connected components* can also be extracted using morphological operations. If $Y$ represents a connected component in an image $A$ and a point $p$ in $Y$ is known, then the following iterative expression yields all the elements of $Y$:

$$X_k = (X_{k-1} \oplus B) \cap A, \text{for } k = 1, 2, \ldots$$

where $X_0 = p$ and $B$ is a $3 \times 3$ matrix of 1s. If $X_k = X_{k-1}$ the algorithm has converged and we let $Y = X_k$.

An important step in representing the structural shape of a planar region is to reduce it to a graph. This is very commonly used in robot path planning. This reduction is most commonly achieved by reducing the region to its *skeleton*.

The skeleton of a region is defined by the medial axis transformation (MAT). The MAT of a region $R$ with border $B$ is defined as follows: for each point $p$ in $R$, we find its closest neighbour in $B$. If $p$ has more than one such closest neighbour, then $p$ belongs to the medial axis (or skeleton) of $R$. Of course, closest depends on the metric used. Figure 9 shows some examples with the usual Euclidean metric.

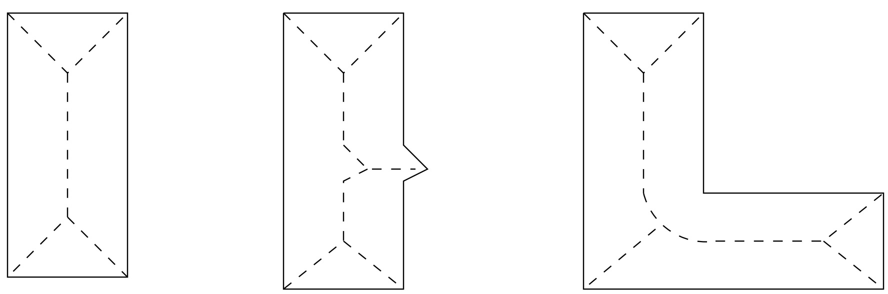

Figure 9: The skeletons of three simple regions.

Direct implementation of the MAT is computationally prohibitive. However, the skeleton of a set can be expressed in terms of erosions and openings. Thus, it can be shown that

$$S(A) = \bigcup_{k=0}^{K} S_k(A),$$

where

$$S_k(A) = (A \ominus kB) - [(A \ominus kB) \circ B],$$

$B$ is a structuring element, $(A \ominus kB)$ indicates $k$ successive erosions of $A$, and $K$ is the last iterative step before $A$ erodes to an empty set.

Thus $A$ can be reconstructed from its skeleton subsets $S_k(A)$ using the equation

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB),$$

where $S_k(A) \oplus kB$ represents $k$ successive dilations of $S_k(A)$.

# References


[1] Rafael C. Gonzalez and Richard E. Woods. *Digital Image Processing*. Addison-Wesley Publishing Company, 1992.

[2] Berthold Klaus Paul Horn. *Robot Vision*. MIT Press, 1986.

[3] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, 1982.